

SWEEPING AT THE MARTIN BOUNDARY OF A FINE DOMAIN

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ABSTRACT. We study sweeping on a subset of the Riesz-Martin space of a fine domain in \mathbb{R}^n ($n \geq 2$), both with respect to the natural topology and the minimal-fine topology, and show that the two notions of sweeping are identical.

1. INTRODUCTION

The fine topology on an open set $\Omega \subset \mathbb{R}^n$ was introduced by H. Cartan in classical potential theory. It is defined as the smallest topology on Ω in which every superharmonic function on Ω is continuous. Potential theory on a finely open set, for example in \mathbb{R}^n , was introduced and studied in the 1970's by the second named author [8]. The harmonic and superharmonic functions and the potentials in this theory are termed finely [super]harmonic functions and fine potentials. Generally one distinguishes by the prefix 'fine(ly)' notions in fine potential theory from those in classical potential theory on a usual (Euclidean) open set. Large parts of classical potential theory have been extended to fine potential theory.

The integral representation of positive finely superharmonic (=nonnegative) functions by using Choquet's method of extreme points was studied by the first named author in [5], where it was shown that the cone of positive superharmonic functions equipped with the natural topology has a compact base. This allowed the present authors in [6] to define the Martin compactification and the Martin boundary of a fine domain U in \mathbb{R}^n . The Martin compactification \overline{U} of U was defined by injection of U in a compact base of the cone $\mathcal{S}(U)$ of positive finely superharmonic functions on U . While the Martin boundary of a usual domain is closed and hence compact, all we can say in the present setup is that the Martin boundary $\Delta(U)$ of U is a G_δ subset of the compact Riesz-Martin space $\overline{U} = U \cup \Delta(U)$ endowed with the natural topology. Nevertheless we can define a suitably measurable Riesz-Martin kernel $K : U \times \overline{U} \rightarrow [0, +\infty]$. Every function $u \in \mathcal{S}(U)$ has an integral representation $u(x) = \int_{\overline{U}} K(x, Y) d\mu(Y)$ in terms of a Radon measure μ on \overline{U} . This representation is unique if it is required that μ be carried by $U \cup \Delta_1(U)$, where $\Delta_1(U)$ denotes the minimal Martin boundary of U , which likewise is a G_δ in \overline{U} . In this case of uniqueness we write $\mu = \mu_u$. We show that u is a fine

potential, resp. an invariant function, if and only if μ_u is carried by U , resp. by $\Delta(U)$. The invariant functions, likewise studied in [6], generalize the positive harmonic functions in the classical Riesz decomposition theorem. Finite valued invariant functions are the same as positive finely harmonic functions.

There is a notion of minimal thinness of a set $E \subset U$ at a point $Y \in \Delta_1(U)$, and an associated minimal-fine filter $\mathcal{F}(Y)$, which allowed the authors in [6] to obtain a generalization of the classical Fatou-Naïm-Doob theorem. We showed that, for any finely superharmonic function $u \geq 0$ on U and for μ_1 -almost every point $Y \in \Delta_1(U)$, $u(x)$ has the limit $(d\mu_u/d\mu_1)(Y)$ as $x \rightarrow Y$ along the minimal-fine filter $\mathcal{F}(Y)$. Here $d\mu_u/d\mu_1$ denotes the Radon-Nikodým derivative of the absolutely continuous component of μ_u with respect to the absolutely continuous component of the measure μ_1 representing the constant function 1, which is finely harmonic and hence invariant.

In the present continuation of [6] we study sweeping on a subset of the Riesz-Martin space, and the Dirichlet problem at the Martin boundary of U . An important integral representation of swept functions (Theorem 3.10) seems to be new even in the case where U is a Euclidean domain. Furthermore we define the notion of minimal thinness of a subset of U at a point of $\Delta_1(U)$, and the associated minimal-fine topology on \overline{U} . This mf-topology is finer than the natural topology on \overline{U} , and induces on U the fine topology there.

In a further continuation [7] of [6] we adapt the PWB method to the study of the Dirichlet problem at the Martin boundary of the fine domain U .

Notations: For a Green domain Ω in \mathbb{R}^n , $n \geq 2$, we denote by G_Ω the Green kernel for Ω . If U is a fine domain in Ω we denote by $\mathcal{S}(U)$ the convex cone of positive finely superharmonic functions on U in the sense of [8]. The convex cone of fine potentials on U (that is, the functions in $\mathcal{S}(U)$ for which every finely subharmonic minorant is ≤ 0) is denoted by $\mathcal{P}(U)$. The cone of invariant functions on U is denoted by $\mathcal{H}_i(U)$; it is the orthogonal band to $\mathcal{P}(U)$ relative to $\mathcal{S}(U)$. By G_U we denote the (fine) Green kernel for U , cf. [9], [10]. If $A \subset U$ and $f : A \rightarrow [0, +\infty]$ one denotes by R_f^A , resp. \widehat{R}_f^A , the reduced function, resp. the swept function, of f on A relative to U , cf. [8, Section 11]. If $u \in \mathcal{S}(U)$ and $A \subset \overline{U}$ we may write \widehat{R}_u^A for \widehat{R}_f^A with $f := 1_A u$. For any set $A \subset \Omega$ we denote by \widetilde{A} the fine closure of A in Ω , and by $b(A)$ the base of A in Ω , that is, the set of points of Ω at which A is not thin, in other words the set of all fine limit points of A in Ω .

2. SWEEPING ON SUBSETS OF \overline{U}

We shall need an ad hoc concept of a (fine) Perron family. Recall from [6, Section 3] the continuous affine form $\Phi \geq 0$ on $\mathcal{S}(U)$ such that the chosen compact base B of the cone $\mathcal{S}(U)$ consists of all $u \in \mathcal{S}(U)$ with $\Phi(u) = 1$.

Cover Ω by a sequence of Euclidean open balls B_k with closures \overline{B}_k contained in Ω . We refer to [6, Lemma 3.14] for the proof of the following lemma:

Lemma 2.1. (a) *The mapping $U \ni y \mapsto G_U(., y) \in \mathcal{S}(U)$ is continuous from U with the fine topology into $\mathcal{S}(U)$ with the natural topology.*

(b) *The function $U \ni y \mapsto \Phi(G_U(., y)) \in]0, +\infty[$ is finely continuous on U .*

(c) *The sets*

$$V_k = \{y \in U : \Phi(G_U(., y)) > 1/k\} \cap B_k$$

form a countable cover of U by finely open sets which are relatively naturally compact in U .

Definition 2.2. A nonvoid lower directed family $\mathcal{F} \subset \mathcal{S}(U)$ is called a (fine) Perron family if $\widehat{R}_u^{U \setminus V_k} \in \mathcal{F}$ for every k and every $u \in \mathcal{F}$.

Theorem 2.3. *If $\mathcal{F} \subset \mathcal{S}(U)$ is a Perron family then $\widehat{\inf} \mathcal{F}$ is an invariant function, and $\widehat{\inf} \mathcal{F} = \inf \mathcal{F}$ on $\{\inf \mathcal{F} < +\infty\}$.*

Proof. Fix k . Clearly

$$\widehat{\inf} \mathcal{F} = \widehat{\inf} \{\widehat{R}_u^{U \setminus V_k} : u \in \mathcal{F}\},$$

and the family $\{\widehat{R}_u^{U \setminus V_k} : u \in \mathcal{F}\}$ is lower directed in U . By [6, Lemma 2.4] each $\widehat{R}_u^{U \setminus V_k}$ is invariant in V_k , and so is therefore $\widehat{\inf} \mathcal{F}|_{V_k}$ according to [6, Theorem 2.6 (c)]. Consequently, $\widehat{\inf} \mathcal{F}$ is likewise invariant, by [6, Theorem 2.6 (b)]. For given $x \in \{\inf \mathcal{F} < +\infty\}$ (if any), choose $u \in \mathcal{F}$ with $u(x) < +\infty$ and an index k so that $x \in V_k$. Then $\widehat{R}_u^{U \setminus V_k} \in \mathcal{F}$. The restriction of $\widehat{R}_u^{U \setminus V_k}$ to the relatively open and hence finely open subset $U \setminus \overline{V}_k$ of U (cf. [6, Corollary 3.15]) is invariant according to [6, Lemma 2.4]. We have $\widehat{R}_u^{U \setminus \overline{V}_k} = u$ on $U \setminus \overline{V}_k$ by [8, Lemma 11.10] because finely open sets are subbasic. It follows that $\widehat{R}_u^{U \setminus V_k} = u$, and so u is invariant on V_k , as noted above. So is therefore every minorant of u in \mathcal{F} , and we conclude from [8, c), p. 132] that indeed $\widehat{\inf} \mathcal{F} = \inf \mathcal{F}$ at the given point x . \square

We are now prepared to study sweeping on \overline{U} , following in part the classical procedure, cf. [4], [2, Section 8.2], the main deviations being caused by the non-compactness of $\Delta(U)$. See also Definition 3.14 and Theorem 3.17 below for the analogous and actually equivalent notion of sweeping relative to the minimal-fine topology on \overline{U} .

Definition 2.4. Let $A \subset \overline{U}$. For any function $u \in \mathcal{S}(U)$ the reduction of u on A is defined by

$$R_u^A = \inf\{v \in \mathcal{S}(U) : v \geq u \text{ on } A \cap U \text{ and on } W \cap U \text{ for some } W \in \mathcal{W}(A)\},$$

where $\mathcal{W}(A)$ denotes the family of all open sets $W \subset \overline{U}$ with the natural topology such that $W \supset A \cap \Delta(U)$. The sweeping \widehat{R}_u^A of u on A is defined as the regularization of R_u^A , that is, the greatest finely l.s.c. minorant of R_u^A .

Thus \widehat{R}_u^A is of class $\mathcal{S}(U)$. It is convenient to express R_u^A and \widehat{R}_u^A in terms of reduction and sweeping on subsets of U , cf. [4, 1.III.5]:

$$R_u^A = \inf \{ R_u^{(A \cup W) \cap U} : W \in \mathcal{W}(A) \},$$

$$\widehat{R}_u^A = \widehat{\inf} \{ \widehat{R}_u^{(A \cup W) \cap U} : W \in \mathcal{W}(A) \}.$$

In particular, for any subset A of U , the present reduction R_u^A and sweeping \widehat{R}_u^A relative to \overline{U} reduce to the similarly denoted usual reduction and sweeping on A relative to U . Note that if $A \subset \Delta(U)$ we may replace $A \cup W$ by W in the above expressions for R_u^A and \widehat{R}_u^A .

By the fundamental convergence theorem [8, Theorem 11.8] and the quasi-Lindelöf property for finely u.s.c. functions (cf. [8, §3.9] for finely l.s.c. functions), there is a decreasing sequence (W_j) of sets $W_j \in \mathcal{W}(A)$ (depending on u) such that it suffices to take for W the sets W_j , in the above definitions and alternative expressions.

Remark 2.5. If $A \subset \Delta(U)$ then $\mathcal{W}(A)$ is the family of all open sets $W \subset \overline{U}$ containing A , and it then suffices to take for W a decreasing sequence of open sets $W_j \supset A$ (depending on u) such that $\bigcap_j \overline{W_j} \subset \overline{A}$. In fact, \overline{A} is the intersection of a decreasing sequence of open sets $V_j \subset \overline{U}$, and we merely have to replace the above (W_j) by the decreasing sequence of open sets $W_j \cap V_j \in \mathcal{W}(A)$ whose intersection clearly is contained in \overline{A} . If A is a compact subset of $\Delta(U)$ we may therefore take $W_j = V_j$ (independently of u).

Proposition 2.6. *Let A and B be two subsets of \overline{U} and let $u, v \in \mathcal{S}(U)$ and $0 < \alpha < +\infty$. Then*

1. $\widehat{R}_u^{A \cup B} \leq \widehat{R}_u^A + \widehat{R}_u^B$.
2. If $A \subset B$ then $\widehat{R}_u^A \leq \widehat{R}_u^B$.
3. If 0 times $+\infty$ is defined to be 0 then $\widehat{R}_{\alpha u}^A = \alpha \widehat{R}_u^A$.
4. $\widehat{R}_{u+v}^A = \widehat{R}_u^A + \widehat{R}_v^A$.
5. For any decreasing sequence of functions $u_j \in \mathcal{S}(U)$ we have $\widehat{\inf}_j \widehat{R}_{u_j}^A = \widehat{R}_{\inf_j u_j}^A$.
6. If $A \subset B$ then $\widehat{R}_{\widehat{R}_u^A}^B = \widehat{R}_{\widehat{R}_u^A}^A = \widehat{R}_u^A$.

Proof. Property 1. is established just as in [4, (4.1), p. 39] (with $v = 0$): For $u_A \in \mathcal{S}(U)$ with $u_A \geq u$ on $A \cup W_A$ for some $W_A \in \mathcal{W}(A)$ and analogous u_B, W_B we have $u_A + u_B \in \mathcal{S}(U)$ and $u_A + u_B \geq u$ on $A \cup B \cup W_A \cup W_B$ with $W_A \cup W_B \in \mathcal{W}(A \cup B)$. This implies $R_u^{A \cup B} \leq R_u^A + R_u^B$. The asserted

inequality therefore holds quasieverywhere and hence everywhere on U , by fine continuity. Property 2. follows from $\mathcal{W}(A) \supset \mathcal{W}(B)$. Property 3. follows from $\widehat{R}_{\alpha u}^{(A \cup W) \cap U} = \alpha \widehat{R}_u^{(A \cup W) \cap U}$ by taking $\widehat{\inf}$ over $W \in \mathcal{W}(A)$. As to 4. we have for any $W \in \mathcal{W}(A)$ $\widehat{R}_{u+v}^{(A \cup W) \cap U} = \widehat{R}_u^{(A \cup W) \cap U} + \widehat{R}_v^{(A \cup W) \cap U}$, whence the asserted equation by taking the natural limits of the decreasing nets on $\mathcal{S}(U)$ in question as the index W ranges over the lower directed family $\mathcal{W}(A)$, cf. [6, Theorem 2.9].

Concerning 5., according to [6, Lemmas 2.1 and 2.3], u_j is Euclidean Borel measurable and

$$R_{u_j}^{(A \cup W) \cap U}(x) = \int_U u_j d\varepsilon_x^{\mathbb{C}(U \setminus (A \cup W))} \leq u_j(x) \leq u_1(x)$$

(complements relative to Ω). For given $x \in U$ with $u_1(x) < +\infty$ and $W \in \mathcal{W}(A)$ consider the equalities

$$\begin{aligned} \inf_j R_{u_j}^{(A \cup W) \cap U}(x) &= \inf_j \int_U u_j d\varepsilon_x^{\mathbb{C}(U \setminus (A \cup W))} = \int_U \inf_j u_j d\varepsilon_x^{\mathbb{C}(U \setminus (A \cup W))} \\ &= \int_U \widehat{\inf_j} u_j d\varepsilon_x^{\mathbb{C}(U \setminus (A \cup W))} = R_{\widehat{\inf_j} u_j}^{(A \cup W) \cap U}(x). \end{aligned}$$

The first and the last equalities hold by [6, Lemma 2.3]. The second equality is obvious (Lebesgue), the integrals being finite by hypothesis. The third equality holds if $\widehat{\inf_j} u_j(x) = \inf_j u_j(x)$, for either $x \in U \cap b((A \cup W) \cap U)$, and then $\varepsilon_x^{\mathbb{C}(U \setminus (A \cup W))} = \varepsilon_x$; or else $x \in U \setminus \widehat{b((A \cup W) \cap U)}$, and then $\varepsilon_x^{\mathbb{C}(U \setminus (A \cup W))}$ does not charge the polar set $\{\inf_j u_j \neq \widehat{\inf_j} u_j\}$. The resulting equality in the above display thus holds q.e. for $x \in U$, and hence also everywhere on U after finely l.s.c. regularization of both members.

Property 6. is known for $A, B \subset U$. For general $A, B \subset \overline{U}$, say with $A \subset B$, the first and the second member of the equalities in 6. lie between $\widehat{R}_{\widehat{R}_u^A}^A$ and \widehat{R}_u^A in view of 2., and it therefore suffices to consider the case where $A = B$. For given $W \in \mathcal{W}(A)$ consider a decreasing sequence $(W_j) \subset \mathcal{W}(A)$ such that $\widehat{R}_u^A = \widehat{\inf_j} \widehat{R}_u^{(A \cup W_j) \cap U}$. Replacing W_j by $W_j \cap W$ we achieve that $W_j \subset W$, and hence

$$\widehat{R}_{\widehat{R}_u^{(A \cup W_j) \cap U}}^{(A \cup W) \cap U} = \widehat{R}_u^{(A \cup W_j) \cap U}.$$

According to 5. this implies 6. by taking $\widehat{\inf_j}$ and next taking $\widehat{\inf}_{W \in \mathcal{W}(A)}$. \square

Proposition 2.7. *Let $u \in \mathcal{S}(U)$. For any subset A of $\Delta(U)$ the function \widehat{R}_u^A is invariant, and we have $\widehat{R}_u^A \preceq u$.*

Proof. Consider the family

$$\mathcal{F} := \{\widehat{R}_u^{W \cap U} : W \in \mathcal{W}(A)\}.$$

Clearly, \mathcal{F} is lower directed. Consider the compact sets $A_{kl} \subset U$ in the proof of [6, Proposition 3.10]. Fix k and V_k from Lemma 2.1. In view of that lemma and the text preceding it, $\overline{U} \setminus A_{kk}$ is open in \overline{U} and contains $\Delta(U)$. In Definition 2.4 of \widehat{R}_u^A it therefore suffices to consider open sets $W \supset A$ such that $W \subset \overline{U} \setminus A_{kk}$, whereby $W \cap U \subset U \setminus A_{kk} \subset U \setminus V_k$. By 6. in Proposition 2.6 we then have $\widehat{R}_{\widehat{R}_u^{W \cap U}}^{U \setminus V_k} = \widehat{R}_u^{W \cap U}$. The lower directed family $\mathcal{F} := \{\widehat{R}_u^{W \cap U} : W \in \mathcal{W}(A)\}$ is therefore a Perron family in the sense of Definition 2.2. By Definition 2.4 we have $\widehat{R}_u^A = \inf \mathcal{F}$, and it therefore follows by Theorem 2.3 that \widehat{R}_u^A indeed is invariant. Consequently, $\widehat{R}_u^A \preccurlyeq u$ in view of [6, Lemma 2.2]. \square

Proposition 2.8. *Let $u \in \mathcal{S}(U)$. (a) For any increasing sequence (A_j) of subsets of \overline{U} we have $\widehat{R}_u^{\bigcup_j A_j} = \sup_j \widehat{R}_u^{A_j}$.*

(b) For any sequence (A_j) of subsets of \overline{U} we have $\widehat{R}_u^{\bigcup_j A_j} \leq \sum_j \widehat{R}_u^{A_j}$.

Proof. For (a) we proceed much as in [4, p. 74, Proof of (e)] (where U is a Euclidean Green domain). Writing $A = \bigcup_j A_j$ and $v = \sup_j \widehat{R}_u^{A_j}$ the inequality $v \leq \widehat{R}_u^A$ is obvious. For the opposite inequality we shall also consider $\widehat{R}_u^{A_j \cap \Delta(U)}$. Consider a point $x \in U$ for which $u(x) < +\infty$ and $\widehat{R}_u^{A_j \cap \Delta(U)}(x) = R_u^{A_j \cap \Delta(U)}(x)$. For any integer $j > 0$ there exists $W_j \in \mathcal{W}(A_j \cap \Delta(U)) = \mathcal{W}(A_j)$ and $v_j \in \mathcal{S}(U)$ such that $v_j \geq u$ on $W_j \cap U$ and

$$v_j(x) \leq R_u^{A_j \cap \Delta(U)}(x) + 2^{-j} = \widehat{R}_u^{A_j \cap \Delta(U)}(x) + 2^{-j}.$$

The swept function $\widehat{R}_u^{A_j \cap \Delta(U)}$ is invariant by Proposition 2.7, and $\widehat{R}_u^{A_j \cap \Delta(U)} \leq \widehat{R}_u^{W_j \cap U} \leq R_u^{W_j \cap U} \leq v_j$. Hence $\widehat{R}_u^{A_j \cap \Delta(U)} \preccurlyeq v_j$. We show that for any integer $k > 0$ the function

$$(2.1) \quad u'_k := v + \sum_{j \geq k} (v_j - \widehat{R}_u^{A_j \cap \Delta(U)})$$

is of class $\mathcal{S}(U)$. In the first place, each term in the sum is of class $\mathcal{S}(U)$. Because v_j is finely continuous and $\widehat{R}_u^{A_j \cap \Delta(U)}$ is finely u.s.c. there is a fine neighborhood V of x with Euclidean compact closure \overline{V} in Ω contained in U and such that $v_j \leq R_u^{A_j \cap \Delta(U)} + 2^{1-j}$ on V and hence on \overline{V} , by fine continuity. We may further arrange that u is bounded on \overline{V} and that $\widehat{R}_u^{A_j \cap \Delta(U)} = \widehat{R}_u^{W_j \cap U}$ on \overline{V} . Then

$$\int (v_j - \widehat{R}_u^{A_j \cap \Delta(U)}) d\varepsilon_x^{\Omega \setminus V} \leq v_j(x) - \widehat{R}_u^{A_j \cap \Delta(U)}(x) \leq 2^{1-j},$$

since $\varepsilon_x^{\Omega \setminus V}$ is carried by \overline{V} and does not charge any polar set. See also [8, Section 8.4]. It follows that the finely hyperharmonic sum in (2.1) is of class $\mathcal{S}(U)$, having a finite integral with respect to $\varepsilon_x^{\Omega \setminus V}$. For any $W \in \mathcal{W}(A_j)$ we have

$\widehat{R}_u^{(A_j \cup W) \cap U} = u$ q.e. on $(A_j \cup W) \cap U$, in particular q.e. on $A_j \cap U$. By Definition 2.4 we have $\widehat{R}_u^{A_j} = u$ q.e. on $A_j \cap U$ (because it suffices to consider a suitable sequence of sets W). It follows that $v = u$ q.e. on each $A_j \cap U$ and hence also q.e. on $A \cap U$. Choose a superharmonic function $s > 0$ on Ω such that $s(y) = +\infty$ for every y in the polar set $\{y \in A \cap U : v(y) \neq u(y)\} \cup \bigcup_{j>0} \{y \in A \cap U : \widehat{R}_u^{A_j \cap \Delta(U)} \neq R_u^{A_j \cap \Delta(U)}\}$. For any $\delta > 0$ we then have $u'_k + \delta s \geq v + \delta s \geq u$ on $A \cap U$. Because $v \geq \widehat{R}_u^{A_j} \geq \widehat{R}_u^{A_j \cap \Delta(U)}$ we obtain for $j \geq k$

$$u'_k \geq \widehat{R}_u^{A_j \cap \Delta(U)} + (v_j - \widehat{R}_u^{A_j \cap \Delta(U)}) = v_j,$$

and hence $u'_k \geq v_j \geq u$ on $W_j \cap U$ for $j \geq k$. Altogether, $u'_k + \delta s \geq u$ on $A \cap U$ and on $W_j \cap U$. It follows by Definition 2.4 that $u'_k + \delta s \geq \widehat{R}_u^A$, and hence for $\delta \rightarrow 0$ that $u'_k \geq \widehat{R}_u^A$ q.e. and actually everywhere on U . But $u'_k \searrow v$ q.e. (namely at each point where u'_1 is finite). Consequently $v \geq \widehat{R}_u^A$ q.e. on U and so indeed everywhere on U .

(b) is easily deduced from (a) applied with A_j replaced by $A_1 \cup \dots \cup A_j$, in view of 1. in Proposition 2.6 (extended to finite unions). There is also a simple direct proof, cf. [2, Lemma 8.2.2 (i)] for the case of a Euclidean Green domain U . \square

Proposition 2.9. *For any $A \subset \overline{U}$ we have $\widehat{R}_u^A = \widehat{R}_u^{A \cap \Delta(U)} + \widehat{R}_v^{A \cap U}$, where $v := u - \widehat{R}_u^{A \cap \Delta(U)}$.*

Proof. Let $s \in \mathcal{S}(U)$, $s \geq u$ on $A \cap U$ and on a neighborhood of $A \cap \Delta(U)$. Then $s \geq \widehat{R}_u^{A \cap \Delta(U)}$, which is invariant by Proposition 2.7, and so $\widehat{R}_u^{A \cap \Delta(U)} \preceq s$. Furthermore, $s - \widehat{R}_u^{A \cap \Delta(U)} \geq u - \widehat{R}_u^{A \cap \Delta(U)} = v$ on $A \cap U$. It follows that $s - \widehat{R}_u^{A \cap \Delta(U)} \geq \widehat{R}_v^{A \cap U}$, and so $s \geq \widehat{R}_u^{A \cap \Delta(U)} + \widehat{R}_v^{A \cap U}$. This shows that $\widehat{R}_u^A \geq \widehat{R}_u^{A \cap \Delta(U)} + \widehat{R}_v^{A \cap U}$. For the opposite inequality let $w \in \mathcal{S}(U)$, $w \geq u$ on $A \cap U$ and on a neighborhood of $A \cap \Delta(U)$. Then $w \geq \widehat{R}_u^{A \cap \Delta(U)}$ on $A \cap U$, and since $\widehat{R}_u^{A \cap \Delta(U)}$ is invariant as noted above, we have $w - \widehat{R}_u^{A \cap \Delta(U)} \in \mathcal{S}(U)$. By hypothesis this function majorizes v on $A \cap U$, and we therefore get $w \geq \widehat{R}_u^{A \cap \Delta(U)} + \widehat{R}_v^{A \cap U}$. By varying w this leads to the remaining inequality $\widehat{R}_u^A \geq \widehat{R}_u^{A \cap \Delta(U)} + \widehat{R}_v^{A \cap U}$. \square

For any (positive Radon) measure μ on \overline{U} we write for brevity $K\mu = \int_{\overline{U}} K(\cdot, Y) d\mu(Y)$. We say that a measure μ represents a function $u \in \mathcal{S}(U)$ if $u = K\mu$.

Corollary 2.10. *Let H be a compact subset of \overline{U} and μ a Radon measure on \overline{U} carried by H . Then $K\mu$ is invariant on $U \setminus H$.*

Proof. By Proposition 2.9 there is a function $v \in \mathcal{S}(U)$ such that $\widehat{R}_{K\mu}^H = \widehat{R}_{K\mu}^{H \cap \Delta(U)} + \widehat{R}_v^{H \cap U}$. By Proposition 2.7 the former term on the right is invariant, and the latter term is invariant on $U \setminus H$ according to [6, Lemma 2.4]. \square

Corollary 2.11. *Let $A \subset \overline{U}$ and $u \in \mathcal{S}(U)$. Then \widehat{R}_u^A from Definition 2.4 is invariant on $U \setminus \overline{A}$.*

Proof. For $A \subset U$ this follows from [6, Lemma 2.4], and for $A \subset \Delta(U)$ it follows from Proposition 2.7. For general $A \subset \overline{U}$ it therefore follows right away by application of Proposition 2.9. \square

For a (positive) Borel measure μ on U we denote by μ_\cdot and μ^\cdot the inner and the outer μ -measure, respectively.

Proposition 2.12. *Let $p = G_U \mu$ be a fine potential on U and let V be a finely open subset of U . Then we have the bi-implications*

$$p|_V \text{ invariant} \iff \mu_\cdot(V) = 0 \iff \mu^\cdot(V) = 0.$$

Proof. Suppose first that $\mu_\cdot(V) = 0$. For any regular finely open set W with $\widetilde{W} \subset V$ the part of μ on the K_σ set W equals 0, and hence $G_W \mu \equiv 0$. It follows by [10, Lemma 2.6] that

$$p = \int p d\varepsilon_x^{\Omega \setminus W}$$

on U , in particular on V . Fix a point $x_0 \in V$. The finely open sets $W_j := \{x \in V : G_U(x, x_0) > \frac{1}{j}\}$ form a countable cover of V such that $\widetilde{W}_j \subset V$. It therefore follows by [10, Theorem 4.4] (with U replaced by V and s by $p|_V$) that $p|_V$ indeed is invariant.

Conversely, suppose that $p|_V$ is invariant, and let us prove that $\mu^\cdot(V) = 0$. Under the extra hypothesis that $\widetilde{V} \subset U$ it now follows by [10, Lemma 2.6] that

$$p = G_V \mu + \widehat{R}_p^{U \setminus V}$$

on U . By [10, Lemma 2.4] the latter term on the right is invariant on V , and so is therefore the difference $p = G_V \mu$, which however is a fine potential on V , and so $G_V \mu = 0$ on V , that is, $\mu(r(V)) = 0$, whence $\mu^\cdot(V) = 0$. Without the above extra hypothesis that $\widetilde{V} \subset U$ we cover V by a sequence of finely open sets

$$W_j := \{x \in V : G_{r(V)}(x, x_0) > \frac{1}{j}\} \subset \{x \in r(V) : G_{r(V)}(x, x_0) \geq \frac{1}{j}\}.$$

The last set is finely closed subset of U , and so $\widetilde{W}_j \subset U$. As shown above, it follows that $\mu^\cdot(W_j) = 0$, and hence indeed $\mu^\cdot(V) = 0$. \square

Proposition 2.13. *Let $A \subset \overline{U}$ and $u \in \mathcal{S}(U)$. Then there exists a measure on \overline{U} representing \widehat{R}_u^A and carried by \overline{A} .*

Proof. We may suppose that $\widehat{R}_u^A \neq 0$, in particular $u > 0$, the case $\widehat{R}_u^A = 0$ being trivial. For any probability measure ν on B we denote in this proof by $b(\nu)$ the barycenter of ν .

Suppose first that $A \subset U$. Let p be a fine potential > 0 on U . For any natural number k there exists a non-zero Radon measure σ_k on \overline{U} representing the fine potential $\widehat{R}_{u \wedge kp}^A > 0$ on U , and σ_k is carried by U according to [6, Corollary 3.25]. In view of the first paragraph of [6, Section 3] we have

$$\widehat{R}_{u \wedge kp}^A = K\sigma_k = \int_U K(., y) d\sigma_k(y) = \int_U G_U(., y) d\tau_k(y)$$

with $d\tau_k(y) = \Phi(G_U(., y))^{-1} d\sigma_k(y)$. Here we use that the finite non-zero function $y \mapsto \Phi(G_U(., y))$ on U is finely continuous by Lemma 2.1 (b) and hence Borel measurable by [6, Lemma 2.1]. Thus there is indeed a non-zero Borel measure τ_k on U as stated. By Corollary 2.11 $\widehat{R}_{u \wedge kp}^A$ is invariant on $U \setminus \overline{A}$, and hence τ_k is carried by \overline{A} according to Proposition 2.12. It follows that σ_k likewise is carried by \overline{A} .

Consider for each k the probability measure ν_k on the chosen compact base B of the cone $\mathcal{S}(U)$, defined by $\nu_k(E) = \sigma_k(E \cap U)/\sigma_k(U)$ for any Borel subset E of \overline{U} . Clearly, ν_k is carried by \overline{A} along with σ_k . The sequence (ν_k) has a subsequence (ν_{k_j}) which converges vaguely to a probability measure ν on \overline{U} , necessarily carried by \overline{A} . On the other hand, $\widehat{R}_{u \wedge kp}^A \rightarrow \widehat{R}_u^A$ pointwise and increasingly for $k \rightarrow +\infty$. It follows by [6, Theorem 2.10] that $\widehat{R}_{u \wedge kp}^A \rightarrow \widehat{R}_u^A$ in the natural topology on $\mathcal{S}(U)$ as $k \rightarrow +\infty$, and hence $\Phi(\widehat{R}_{u \wedge kp}^A) \rightarrow \Phi(\widehat{R}_u^A) \in]0, +\infty[$ because Φ is naturally continuous on $\mathcal{S}(U)$. Identifying as usual ν_k and ν with probability measures on B we infer that

$$\frac{1}{\Phi(\widehat{R}_u^A)} \widehat{R}_u^A = \lim_{j \rightarrow \infty} \frac{1}{\Phi(\widehat{R}_{u \wedge k_j p}^A)} \widehat{R}_{u \wedge k_j p}^A = \lim_{j \rightarrow \infty} b(\nu_{k_j}) = b(\nu) = K\nu.$$

Hence $\widehat{R}_u^A = K\mu$, where $\mu := \Phi(\widehat{R}_u^A)\nu$ (now again considered as a measure on \overline{U}) is carried by \overline{A} along with ν .

Next, let $A \subset \Delta(U)$. According to Remark 2.5 there is a decreasing sequence of open sets W_j (depending on u) such that $A \subset \bigcap_j W_j \subset \bigcap_j \overline{W_j} \subset \overline{A}$ and $\widehat{R}_u^A = \widehat{\inf_j \widehat{R}_u^{W_j \cap U}} = \lim_j \widehat{R}_u^{W_j \cap U}$ (natural limit, again by [6, Theorem 2.10]). There is a sequence of reals $\alpha_j > 0$ and a real $\alpha > 0$ such that $\alpha_j \widehat{R}_u^{W_j \cap U} \in B$ and $\alpha \widehat{R}_u^A \in B$. The sequence (α_j) converges to α because the sequence $(\widehat{R}_u^{W_j \cap U})$ converges naturally to \widehat{R}_u^A . For any index j there exists, as shown in the preceding paragraph, a probability measure μ_j on B with the barycenter $\alpha_j \widehat{R}_u^{W_j \cap U}$ such that μ_j (when viewed as a measure on \overline{U}) is carried by $\overline{W_j}$. After passing to a subsequence we may suppose that μ_j converges to a probability

measure μ on B which (again when viewed as a measure on \overline{U}) necessarily is carried by $\bigcap_j \overline{W}_j \subset \overline{A}$. The sequence $(b(\mu_j)) = (\alpha_j \widehat{R}_u^{W_j})$ of barycenters of the μ_j therefore converges to the barycenter $b(\mu)$ of μ , whence $K\mu = b(\mu) = \alpha \widehat{R}_u^A$, and \widehat{R}_u^A is represented by the measure $\frac{1}{\alpha}\mu$ carried by \overline{A} .

In the general case where just $A \subset \overline{U}$ we have by Proposition 2.9 $\widehat{R}_u^A = \widehat{R}_u^{A \cap \Delta(U)} + \widehat{R}_v^{A \cap U}$, where $v \in \mathcal{S}(U)$. As shown in the third paragraph of the present proof there exists a measure μ_1 on \overline{U} representing $\widehat{R}_v^{A \cap U}$ and carried by \overline{A} . And as shown in the preceding paragraph there exists a measure μ_2 on \overline{U} representing $\widehat{R}_u^{A \cap \Delta(U)}$ and likewise carried by \overline{A} . The measure $\mu = \mu_1 + \mu_2$ therefore represents \widehat{R}_u^A and is carried by \overline{A} . \square

Proposition 2.14. *Let $A \subset \overline{U}$. Then*

- (i) *If $A \subset \Delta(U)$ then $\widehat{R}_p^A = 0$ for any $p \in \mathcal{P}(U)$.*
- (ii) *If $A \subset \Delta(U)$ and $Y \in U \cup \Delta_1(U)$ then either $\widehat{R}_{K(.,Y)}^A = 0$ or $\widehat{R}_{K(.,Y)}^A = K(.,Y)$. If moreover $Y \notin A$ then $\widehat{R}_{K(.,Y)}^A = 0$.*
- (iii) *If $Y \in \Delta_1(U) \setminus \overline{A}$ then $\widehat{R}_{K(.,Y)}^A \neq K(.,Y)$.*

Proof. It follows from Proposition 2.7 that \widehat{R}_p^A and $\widehat{R}_{K(.,Y)}^A$ are invariant. This establishes (i) because \widehat{R}_p^A is also a fine potential (along with p).

For the former assertion (ii) we have $\widehat{R}_{K(.,Y)}^A \preccurlyeq K(.,Y)$, again by Proposition 2.7, and since $K(.,Y)$ is extreme there is a constant $c \geq 0$ such that $\widehat{R}_{K(.,Y)}^A = cK(.,Y)$. Hence it follows from 6. in Proposition 2.6 with $A = B$ that $c = 0$ or $c = 1$.

For the latter assertion (ii) suppose first that $Y \notin \overline{A}$ (the natural closure of A in \overline{U}). Suppose by contradiction that $\widehat{R}_{K(.,Y)}^A = K(.,Y)$. According to Proposition 2.13 there exists a measure λ on \overline{U} carried by \overline{A} such that

$$(2.2) \quad \widehat{R}_{K(.,Y)}^A = \int_{\overline{U}} K(.,Z) d\lambda(Z).$$

It follows that $K(.,Y) = \int_{\overline{U}} K(.,Z) d\lambda(Z)$, and so λ is a probability measure. Denote μ the probability measure on B corresponding to λ under the identification of B with $\{K(.,Z) : Z \in \overline{U}\}$. Then μ has the barycenter $K(.,Y)$. Since $K(.,Y)$ is an extreme point of B we infer by [1, Corollary I.2.4, p.15] that $\mu = \varepsilon_Y$, and hence $Y \in \overline{A}$, which is contradictory. Thus $\widehat{R}_{K(.,Y)}^A \neq K(.,Y)$, and consequently $\widehat{R}_{K(.,Y)}^A = 0$ according to the former assertion (ii) above.

It remains to consider the case where we just have $Y \notin A$. In this case A may be written as the union of an increasing sequence of subsets A_j of A with

$Y \notin \overline{A}_j$ for any j . By Proposition 2.8 (b) we then have $\widehat{R}_{K(.,Y)}^A \leq \sum_j \widehat{R}_{K(.,Y)}^{A_j} = 0$, and hence $\widehat{R}_{K(.,Y)}^A = 0$, as claimed.

For (iii), suppose by contradiction that $\widehat{R}_{K(.,Y)}^A = K(.,Y)$. Again, there exists by Proposition 2.13 a probability measure λ on \overline{U} carried by \overline{A} such that (2.2) holds, and hence $Y \in \overline{A}$, which is contradictory. \square

Actually, in Proposition 2.14 (ii), if $Y \in A$ and hence $Y \notin U$ then $Y \in \Delta_1(U)$, and it follows that $\widehat{R}_{K(.,Y)}^A = K(.,Y)$, see Proposition 3.9 below.

The following result extends 4. in Proposition 2.6 to infinite sums.

Proposition 2.15. *Let $A \subset \overline{U}$. Let (μ_j) be a sequence of measures on \overline{U} such that $\sum_j \int d\mu_j < +\infty$, and let $\mu = \sum_j \mu_j$. Then*

$$\widehat{R}_{K\mu}^A = \sum_j \widehat{R}_{K\mu_j}^A.$$

Proof. If $A \subset U$ we have indeed by [6, Lemma 2.3] for $x \in U$

$$\begin{aligned} R_{K\mu}^A(x) &= \int K\mu d\varepsilon_x^{A \cup (\Omega \setminus U)} = \int \left(\int K(.,Y) d\varepsilon_x^{A \cup (\Omega \setminus U)} \right) d\mu(Y) \\ &= \sum_j \int \left(\int K(.,Y) d\varepsilon_x^{A \cup (\Omega \setminus U)} \right) d\mu_j(Y) = \sum_j \int K\mu_j d\varepsilon_x^{A \cup (\Omega \setminus U)} \\ &= \sum_j R_{K\mu_j}^A(x), \end{aligned}$$

the applications of Fubini's theorem being justified by the conclusion of [6, Remark 3.4]. It only remains to perform the l.s.c. regularization of both members of the resulting equation.

Next, let $A \subset \Delta(U)$. The inequality ' \geq ' being trivial we may suppose that the finely hyperharmonic function given by the right hand side of the asserted equation is of class $\mathcal{S}(U)$, viz. $\neq +\infty$. Suppose first that $\widehat{R}_{K\mu}^A = K\mu$. For integers $k > 0$ we have by 4. in Proposition 2.6 (extended to sums of finitely many measures)

$$\widehat{R}_{K\mu}^A = \sum_{j \leq k} \widehat{R}_{K\mu_j}^A + \widehat{R}_{K(\sum_{j > k} \mu_j)}^A.$$

Since $\sum_j K\mu_j = K\mu < +\infty$ q.e. we have

$$\widehat{R}_{K(\sum_{j > k} \mu_j)}^A \leq K \sum_{j > k} \mu_j = \sum_{j > k} K\mu_j \searrow 0$$

q.e. as $k \rightarrow \infty$. We thus have $\widehat{R}_{K\mu}^A \geq \sum_{j > 0} \widehat{R}_{K\mu_j}^A$ with equality q.e., and indeed everywhere, both members of the inequality being of class $\mathcal{S}(U)$.

Without the temporary hypothesis $\widehat{R}_{K\mu}^A = K\mu$ we have $\widehat{R}_{K\mu}^A \preccurlyeq K\mu$ because $\widehat{R}_{K\mu}^A$ is invariant according to Proposition 2.7. Thus there exists a measure ν on \overline{U} with $\nu \leq \mu$ and hence $K\nu \in \mathcal{S}(U)$ such that $K\mu = \widehat{R}_{K\mu}^A + K\nu$. After sweeping on A while invoking 6. in Proposition 2.6 we obtain $\widehat{R}_{K\nu}^A = 0$. Thus $K(\mu - \nu) = \widehat{R}_{K(\mu-\nu)}^A = \widehat{R}_{K\mu}^A$. Similarly, $K\mu_j = \widehat{R}_{K\mu_j}^A + K\nu_j$ with $\nu_j \leq \mu_j$ and $K(\mu_j - \nu_j) = \widehat{R}_{K\mu_j}^A$. As shown above it follows that $\widehat{R}_{K(\mu-\nu)}^A = \sum_j \widehat{R}_{K(\mu_j-\nu_j)}^A$ and hence

$$\widehat{R}_{K\mu}^A = \widehat{R}_{K\nu}^A + \sum_j \widehat{R}_{K\mu_j}^A = \sum_j \widehat{R}_{K\mu_j}^A$$

because $\widehat{R}_{K\nu}^A = 0$. The general case $A \subset \overline{U}$ follows immediately by application of Proposition 2.9. \square

Remark 2.16. Even in the classical case $U = \Omega$, sweeping (and reduction) of a function $u \in \mathcal{S}(U)$ on an arbitrary set $A \subset \Delta(U)$ lacks the following two properties, valid when $A \subset U$. Fix a point $Y \in A \cap \Delta_1(U)$ and note that $\widehat{R}_{K(\cdot, Y)}^A = K(\cdot, Y)$ by Proposition 3.9 below. In the classical case, $K(\cdot, Y) \wedge c$ is a potential (hence a fine potential) for any constant $c > 0$, as noted in [4, Observation, p. 74] for the purpose of showing that the following Property 1. fails when $A = \Delta(U)$:

1. For any increasing sequence of functions $u_j \in \mathcal{S}(U)$ with pointwise supremum $u \in \mathcal{S}(U)$, we should have $\widehat{R}_u^A = \sup_j \widehat{R}_{u_j}^A$. This holds when $A \subset U$, by [8, Theorem 11.12], but fails (classically) for $A = \Delta(U)$ and $u_j = K(\cdot, Y) \wedge j$ in view of the above. It does hold, however, for any sequence $(u_j) \subset \mathcal{S}(U)$ which is increasing in the specific order; this is a reformulation of Proposition 2.15 above.

2. For any $x \in U$, the affine function $u \mapsto \widehat{R}_u^A(x)$ on $\mathcal{S}(U)$ should be (naturally) l.s.c. For the proof that this holds for $A \subset U$ we may assume that A is a base relative to U , and hence $\widehat{R}_u^A = R_u^A$ and ε_x^A is carried by A for any $u \in \mathcal{S}(U)$. Consider a sequence of functions $u_j \in \mathcal{S}(U)$ converging (naturally) to $u \in \mathcal{S}(U)$. Then

$$\begin{aligned} R_u^A(x) &= \int_U u \, d\varepsilon_x^{A \cup (\Omega \setminus U)} = \int_U \liminf_j \widehat{u}_j \, d\varepsilon_x^{A \cup (\Omega \setminus U)} \\ &\leq \int_U \liminf_j u_j \, d\varepsilon_x^{A \cup (\Omega \setminus U)} \leq \liminf_j \int_U u_j \, d\varepsilon_x^{A \cup (\Omega \setminus U)} = \liminf_j R_{u_j}^A(x) \end{aligned}$$

by [6, Lemmas 2.1 and 2.3, and Theorem 2.10], using Fatou's lemma; and it only remains to regularize. But Property 2. fails (classically) for $A = \Delta(U)$ and $u_j = K(\cdot, Y) \wedge j$, hence $u = K(\cdot, Y)$, in view of the above.

3. MINIMAL THINNESS AND THE MINIMAL-FINE TOPOLOGY

The following lemma extends [6, Lemma 4.2], in which $E \subset U$.

Lemma 3.1. *For any set $E \subset \overline{U}$ and any point $Y \in \Delta_1(U)$ we have $\widehat{R}_{K(.,Y)}^E \neq K(.,Y)$ if and only if $\widehat{R}_{K(.,Y)}^E \in \mathcal{P}(U)$ (the fine potentials on U).*

Proof. If $\widehat{R}_{K(.,Y)}^E$ is a fine potential then $\widehat{R}_{K(.,Y)}^E \neq K(.,Y)$ because $K(.,Y)$ is invariant. Conversely, suppose that $\widehat{R}_{K(.,Y)}^E \neq K(.,Y)$, and write $\widehat{R}_{K(.,Y)}^E = p + h$ with p a fine potential and h invariant. Then $h \leq \widehat{R}_{K(.,Y)}^E \leq K(.,Y)$ and hence by [6, Lemma 2.2] $h \preceq K(.,Y)$, which shows that $h = \alpha K(.,Y)$ for some $\alpha \in [0, 1]$. Here $\alpha \neq 1$, for otherwise $(h =) \widehat{R}_{K(.,Y)}^E = K(.,Y)$ contrary to hypothesis. On the other hand it follows by 6. (with $A = B$) and 4. in Proposition 2.6 that

$$\widehat{R}_{K(.,Y)}^E = \widehat{R}_{\widehat{R}_{K(.,Y)}^E}^E = \widehat{R}_{p+h}^E = \widehat{R}_p^E + \widehat{R}_h^E = p + h,$$

whence $\widehat{R}_p^E = p$ and $\widehat{R}_h^E = h$. If $h \neq 0$ then $\alpha \neq 0$ because $h = \alpha K(.,Y)$. Since $h = \widehat{R}_h^E = \alpha \widehat{R}_{K(.,Y)}^E = \alpha p + \alpha h$ we would obtain $(1 - \alpha)h = \alpha p$ with $0 < \alpha < 1$, which is impossible. Thus actually $h = 0$, and so indeed $\widehat{R}_{K(.,Y)}^E = p \in \mathcal{P}(U)$. \square

Definition 3.2. A set $E \subset U$ is said to be minimal-thin at a point $Y \in \Delta_1(U)$ if $\widehat{R}_{K(.,Y)}^E \neq K(.,Y)$, or equivalently if $\widehat{R}_{K(.,Y)}^E \in \mathcal{P}(U)$, that is (by the preceding lemma) if $\widehat{R}_{K(.,Y)}^E \in \mathcal{P}(U)$.

Corollary 3.3. *For any $Y \in \Delta_1(U)$ the sets $E \subset U$ which are minimal-thin at Y form a filter $\mathcal{F}(Y)$ on U .*

This follows from Lemma 3.1 which easily implies that for any $E_1, E_2 \subset U$ such that $\widehat{R}_{K(.,Y)}^{U \setminus E_i} \neq K(.,Y)$ for $i = 1, 2$, we have $\widehat{R}_{K(.,Y)}^{U \setminus (E_1 \cup E_2)} \neq K(.,Y)$.

Like in classical potential theory we define the minimal-fine (mf) topology on \overline{U} as follows:

Definition 3.4. A set $W \subset \overline{U}$ is said to be a minimal-fine neighborhood of a point $Y \in \overline{U}$ if

- (a) $W \cap U$ is a fine neighborhood of Y in the usual sense, in case $Y \in U$,
- (b) W contains the point Y and $U \setminus W$ is minimal-thin at Y , in case $Y \in \Delta_1(U)$,
- (c) W contains the point Y , in case $Y \in \Delta(U) \setminus \Delta_1(U)$.

In the sequel we will denote by mf-lim and mf-liminf the limit and the liminf in the sense of the mf-topology.

According to (a) above, the mf-topology on \overline{U} induces on U the fine topology there, and U is mf-open in \overline{U} , that is, $\Delta(U)$ is mf-closed in \overline{U} (since \emptyset is minimal-thin at any point $Y \in \Delta_1(U)$). According to (c), the mf-topology on \overline{U} induces the discrete topology on $\Delta(U) \setminus \Delta_1(U)$. In view of (c), (b), and Definition 3.2, $\Delta_1(U)$ is the set of mf-limit points of U in \overline{U} .

Proposition 3.5. *The mf-topology on \overline{U} is finer than the natural topology (and is therefore Hausdorff).*

Proof. Let W be a natural neighborhood of a point $Y \in \overline{U}$. If $Y \in U$ then $W \cap U$ is a usual fine neighborhood of Y in U according to Lemma 2.1 (c), and hence an mf-neighborhood of Y in \overline{U} by Definition 3.4 (a) above. If $Y \in \Delta(U) \setminus \Delta_1(U)$ there is nothing to prove in view of (c) in that definition. In the remaining case where $Y \in \Delta_1(U)$ we show that $U \setminus W$ is minimal-thin at Y , cf. Definition 3.4 (b), which by Lemma 3.1 means that $\widehat{R}_{K(.,Y)}^{U \setminus W} \neq K(., Y)$. Suppose that, on the contrary, $\widehat{R}_{K(.,Y)}^{U \setminus W} = K(., Y)$. Writing $U \setminus W = A$ we have $Y \in W \subset \mathbb{C}\overline{A}$ (complement relative to \overline{U}) because W is open. It follows by Proposition 2.14 (iii) that $\widehat{R}_{K(.,Y)}^A \neq K(., Y)$, which is contradictory. \square

Definition 3.6. Let h be a non-zero minimal invariant function. A point $Y \in \overline{U}$ is termed a pole of h if $\widehat{R}_h^{\{Y\}} = h$.

Remark 3.7. Any pole Y of h belongs to $\Delta(U)$, for if $Y \in U$ then $\widehat{R}_h^{\{Y\}} = 0$ because $\{Y\}$ is polar.

Theorem 3.8. *Every non-zero minimal invariant function on U has precisely one pole. For any $Y \in \Delta_1(U)$ the pole of $K(., Y)$ is Y .*

Proof. Recall from [6, Proposition 3.6] (and the beginning of [6, Section 3]) that the non-zero minimal invariant functions on U are precisely the functions of the form $K(., Y)$ for a (unique) $Y \in \Delta_1(U)$. Consider the family \mathcal{C} of all (necessarily nonvoid) compact subsets C of \overline{A} such that $\widehat{R}_{K(.,Y)}^C = K(., Y)$, and note that \mathcal{C} is nonvoid, for $\overline{U} \in \mathcal{C}$ because $\widehat{R}_{K(.,Y)}^{\overline{U}} = K(., Y)$ according to Proposition 2.14 (ii), (iii). Equip \mathcal{C} with the order defined by the inverse inclusion ‘ \supset ’. For any totally ordered subfamily \mathcal{C}' of \mathcal{C} the intersection C' of \mathcal{C}' satisfies $\widehat{R}_{K(.,Y)}^{C'} = K(., Y)$ in view of the fundamental convergence theorem, and hence \mathcal{C} has a minimal element C_0 according to Zorn’s lemma. The natural topology is Hausdorff, so if C_0 contains two distinct points Z_1 and Z_2 then there are compact subsets C_1 and C_2 of C_0 such that $C_0 = C_1 \cup C_2$, $Z_1 \in C_0 \setminus C_2$ and $Z_2 \in C_0 \setminus C_1$. Since $K(., Y)$ is extreme it then follows by Riesz decomposition that either $\widehat{R}_{K(.,Y)}^{C_1} = K(., Y)$ or $\widehat{R}_{K(.,Y)}^{C_2} = K(., Y)$. In other words, either C_1 or C_2 belongs to \mathcal{C} , say $C_1 \in \mathcal{C}$. By minimality of C_0 we would then have $C_1 = C_0$ which contradicts $Z_2 \in C_0 \setminus C_1$. This shows that indeed $C_0 = \{Z\} \in \mathcal{C}$

for a certain $Z \in \overline{U}$, that is $\widehat{R}_{K(\cdot, Y)}^{\{Z\}} = K(\cdot, Y)$. Thus Z is a pole of $K(\cdot, Y)$. Since $\{Z\}$ is a closed set it follows by Proposition 2.13 and Choquet's theorem that $\widehat{R}_{K(\cdot, Y)}^{\{Z\}} = K\mu$ for some probability measure μ on the compact base B of the cone $\mathcal{S}(U)$ such that μ is carried by $\{Z\}$, that is, for $\mu = \varepsilon_Z$. Thus $K(\cdot, Y) = K\mu = K(\cdot, Z)$, and so indeed $Y = Z$. \square

Proposition 3.9. *Let h be a non-zero minimal invariant function on U with pole $Y \in \Delta_1(U)$ and let $A \subset \Delta(U)$. Then $\widehat{R}_h^A = h$ or 0 if $Y \in A$ or $Y \notin A$, respectively.*

Proof. By Proposition 2.7 the function \widehat{R}_h^A is invariant and $\preceq K(\cdot, Y)$, hence of the form $cK(\cdot, Y)$ for some constant c . It follows by 6. in Proposition 2.6 with $A = B$ that $c = 0$ or $c = 1$. If $A \subset \Delta(U)$ contains the pole Y of h then $h \geq \widehat{R}_h^A \geq \widehat{R}_h^{\{Y\}} = h$, and so $\widehat{R}_h^A = h$. The rest follows from Proposition 2.14 (ii). \square

The following integral representation of the sweeping of a function of class $\mathcal{S}(U)$ on arbitrary sets $A \subset \overline{U}$ is based on Proposition 3.9, which in turn depended on Proposition 2.7.

Theorem 3.10. *For any set $A \subset \overline{U}$ and any Radon measure μ on \overline{U} carried by $U \cup \Delta_1(U)$ we have*

$$(3.1) \quad \widehat{R}_{K\mu}^A = \int^* \widehat{R}_{K(\cdot, Y)}^A d\mu(Y).$$

If A is μ -measurable then the upper integral becomes a true integral.

Proof. For any subset A of U this integral representation was established in [6, Lemma 3.21] with the upper integral replaced by the integral. For $A \subset \Delta(U)$ it suffices to consider the case where μ is carried by $\Delta_1(U)$, for if ν denotes the restriction of μ to U then $K\nu$ and $K(\cdot, Y)$ (for $Y \in U$) are fine potentials according to [6, Corollary 3.25], and so $\widehat{R}_{K\nu}^A = \widehat{R}_{K(\cdot, Y)}^A = 0$ by Proposition 2.14 (i).

Proof that the inequality ' \geq ' holds in (3.1) for $A \subset \Delta(U)$. We may assume that $\widehat{R}_{K\mu}^A \neq +\infty$, that is $\widehat{R}_{K\mu}^A \in \mathcal{S}(U)$. By Remark 2.5 there is a decreasing sequence (W_j) of sets of class $\mathcal{W}(A)$ such that it suffices in Definition 2.4 to take for $W \in \mathcal{W}(A)$ the sets W_j . We show that the following equations and inequality hold quasieverywhere on U :

$$\begin{aligned} \widehat{R}_{K\mu}^A &= \inf_1 \widehat{R}_{K\mu}^{W_j \cap U} = \inf_2 \widehat{R}_{K\mu}^{W_j \cap U} = \inf_3 \int \widehat{R}_{K(\cdot, Y)}^{W_j \cap U} d\mu(Y) \\ &= \int \inf_j \widehat{R}_{K(\cdot, Y)}^{W_j \cap U} d\mu(Y) = \int \inf_j \widehat{R}_{K(\cdot, Y)}^{W_j \cap U} d\mu(Y) \geq \int^* \widehat{R}_{K(\cdot, Y)}^A d\mu(Y). \end{aligned}$$

When these relations have been established quasieverywhere on U , the desired resulting inequality holds everywhere on U . In fact, $\widehat{R}_{K\mu}^A \in \mathcal{S}(U)$ along with $K\mu$; and by Proposition 3.9 we have since μ is carried by $\Delta_1(U)$

$$\begin{aligned} \int^* \widehat{R}_{K(\cdot, Y)}^A d\mu(Y) &= \int^* K(\cdot, Y) 1_A(Y) d\mu(Y) \\ &= \int K(\cdot, Y) 1_{A^*}(Y) d\mu(Y) = K(1_{A^*}\mu) \in \mathcal{S}(U), \end{aligned}$$

where $A^* \subset \overline{U}$ denotes a G_δ set containing A such that $\mu^*(A^* \setminus A) = 0$, cf. [6, Theorem 3.20]. Equation 1 and inequality 6 hold everywhere on U by Definition 2.4. Eq. 2 holds quasieverywhere by the fundamental convergence theorem [8, Theorem 11.8]. Eq. 3 holds at any point $x \in U$ at which $K\mu(x) < +\infty$ and hence $\widehat{R}_{K\mu}^A(x) < +\infty$, for there we have by [6, Lemma 3.21] $\widehat{R}_{K\mu}^{W_j \cap U}(x) = \int \widehat{R}_{K(\cdot, Y)}^{W_j \cap U}(x) d\mu(Y)$, which is finite for large j (depending on x). Eq. 4 is obvious (Lebesgue) at points x as stated for eq. 3. In the first place, $\widehat{R}_{K(\cdot, Y)}^{W_j \cap U}$ is of class $\mathcal{S}(U)$ for each $Y \in \Delta_1(U)$, hence finely continuous and in particular Borel measurable on U according to [6, Lemma 2.1]. Secondly, the integrals are finite, being majorized by $\int K(\cdot, Y) d\mu(Y) = K\mu < +\infty$ at points x as stated. Concerning the remaining eq. 5, note that for each k the function $\widehat{R}_{K(\cdot, Y)}^{W_k \cap U}$ is invariant on $U \setminus \widetilde{W}_k$ according to [6, Lemma 2.4]. For any j and any $k \geq j$ we have $W_k \subset W_j$, and $\widehat{R}_{K(\cdot, Y)}^{W_k \cap U}$ is therefore invariant on each $U \setminus \widetilde{W}_j$, and hence on their union according to [6, Theorem 2.6 (a), (b)]. It follows by [6, Theorem 2.6 (c)] that $\widehat{R}_{K(\cdot, Y)}^{W_j \cap U}$ is invariant on the finely open set U . For any point $x \in U$ such that $K\mu(x) < +\infty$ the set

$$E_x : \{Y \in \overline{U} : \widehat{R}_{K(\cdot, Y)}^A(x) = +\infty\}$$

is μ -null. According to [6, Theorem 2.6 (c)] we obtain

$$\widehat{\inf_j \widehat{R}_{K(\cdot, Y)}^{W_j \cap U}}(x) = \inf_j \widehat{R}_{K(\cdot, Y)}^{W_j \cap U}(x) \quad \mu\text{-a.e. for } Y \in \overline{U},$$

which implies eq. 5 at points $x \in U$ with $K\mu(x) < \infty$. We have thus shown that $\widehat{R}_{K\mu}^A \geq \int^* \widehat{R}_{K(\cdot, Y)} d\mu(Y)$ also for $A \subset \Delta(U)$.

The asserted equality in case $A \subset \Delta(U)$. Recall from Lemma 2.1 (c) the countable cover (V_k) of U by finely open sets with natural closures \overline{V}_k in \overline{U} contained in U . The complements $D_k := \mathcal{C}\overline{V}_k$ (relative to \overline{U}) form a decreasing sequence of open subsets of \overline{U} with the intersection $\Delta(U)$, and such that the mf-closures \widetilde{D}_k likewise have the intersection $\Delta(U)$. Consider first the case where $A = C \cap \Delta(U)$, C compact in \overline{U} . Choose a decreasing sequence of open subsets C_k of \overline{U} containing C such that $\bigcap_k \overline{C}_k = C$. Suppose to begin with that μ is carried by some compact set $E \subset \Delta(U) \setminus A = \Delta(U) \setminus C$. We may

assume that $E \cap C_k = \emptyset$. The decreasing open sets $W_k := C_k \cap D_k \supset A$ are of class $\mathcal{W}(A)$ and hence

$$(3.2) \quad \widehat{R}_{K\mu}^A \leq \widehat{R}_{K\mu}^{W_k \cap U} = \int_E \widehat{R}_{K(.,Y)}^{W_k \cap U} d\mu(Y).$$

For each $Y \in E$ the functions $p_k := \widehat{R}_{K(.,Y)}^{W_k \cap U}$ are fine potentials on U according to Lemma 3.1 because the mf-closure \widetilde{W}_k of W_k is contained in $\widetilde{C}_k \cap \widetilde{D}_k \subset \overline{C}_k \cap \widetilde{D}_k$ according to Proposition 3.5, and hence does not meet E and in particular does not contain Y . It follows that $p := \inf_k p_k$ is a fine potential on U . By [6, Lemma 2.4] the restriction of p_k to the finely open set $U \setminus \widetilde{W}_k$ is invariant. By [6, Theorem 2.5] (b) it follows that so is the restriction of p_k to

$$\bigcup_{l \geq k} U \setminus \widetilde{W}_l = U \setminus \bigcap_{l \geq k} \widetilde{W}_l \supset U \setminus \bigcap_{l \geq k} \overline{C}_l \cap \widetilde{D}_l = U \setminus A = U.$$

By [6, Theorem 2.6] (c) we infer that p is itself invariant on U , and being also a fine potential p must be 0. It therefore follows by (3.2) that $\widehat{R}_{K\mu}^A \leq \int_E \widehat{R}_{K(.,Y)}^{W_k \cap U} d\mu(Y) \searrow 0$ pointwise on U , and hence $\widehat{R}_{K\mu}^A = 0 \leq \int \widehat{R}_{K(.,Y)}^A d\mu(Y)$. In combination with the opposite inequality in (3.1) obtained above (now with a true integral) this establishes equality in (3.1) in the present case where $A = C \cap \Delta(U)$ with C compact in \overline{U} and μ carried by a compact set $E \subset \Delta(U) \setminus A$.

Next, replace the latter assumption on μ by the weaker temporary assumption that $\mu(A) = 0$. Choose an increasing sequence of compact sets $E_j \subset \Delta(U) \setminus A$ such that $\mu(E_j) \nearrow \mu(A)$, and denote by μ_j the part of μ on E_j . By Proposition 2.15 it follows that

$$\widehat{R}_{K\mu}^A = \sup_j \widehat{R}_{K\mu_j}^A = \sup_j \int \widehat{R}_{K(.,Y)}^A d\mu_j = \int \widehat{R}_{K(.,Y)}^A d\mu(Y) = 0$$

according to Proposition 2.14 (ii).

Without any such temporary assumption on μ we denote by μ_A and μ' the parts of μ on A and on $\overline{U} \setminus A$, respectively. Then $\mu'(A) = 0$ and hence

$$\widehat{R}_{K\mu}^A = \widehat{R}_{K\mu_A}^A + \widehat{R}_{K\mu'}^A \leq K\mu_A + \int \widehat{R}_{K(.,Y)}^A d\mu'(Y) = \int \widehat{R}_{K(.,Y)}^A d\mu(Y).$$

When combined with the inequality ' \geq ' in (3.1) obtained above (with an upper integral) this leads to equality in (3.1) (with a true integral) for arbitrary μ when $A = C \cap \Delta(U)$ with C compact.

More generally, if $A = C \cap \Delta(U)$ and if C is just the union of an increasing sequence of compact sets $C_j \subset \overline{U}$, then

$$\widehat{R}_{K\mu}^{C_j \cap \Delta(U)} = \int \widehat{R}_{K(.,Y)}^{C_j \cap \Delta(U)} d\mu(Y) \leq \int \widehat{R}_{K(.,Y)}^A d\mu(Y).$$

For $j \rightarrow \infty$ it follows by (a) in Proposition 2.8 that $\widehat{R}_{K\mu}^A \leq \int \widehat{R}_{K(.,Y)}^A d\mu(Y)$. Together with the opposite inequality obtained above this leads to (3.1) (with a true integral) for any set $A = C \cap \Delta(U)$ with C a K_σ subset of \overline{U} . This applies in particular to $A = C \cap \Delta(U)$ with C open in \overline{U} .

Next, let A be any G_δ subset of $\Delta(U)$. Since $\Delta(U)$ is itself a G_δ in \overline{U} this means that $A = C \cap \Delta(U)$ for some G_δ subset C of \overline{U} . Thus C is the intersection of a decreasing sequence of open sets $C_j \subset \overline{U}$. Denote by μ_j the part of μ on $\Delta(U) \setminus A_j$. Then $\mu_j(A_j) = 0$, and again, since A_j is a K_σ subset of $\Delta(U)$,

$$\widehat{R}_{K\mu_j}^A \leq \widehat{R}_{K\mu_j}^{A_j} = \int \widehat{R}_{K(.,Y)}^{A_j} d\mu_j(Y) = 0$$

according to Proposition 2.14 (ii). But $\widehat{R}_{K\mu_j}^A \nearrow \widehat{R}_{K\mu}^A = 0$ in the specific order by Proposition 2.15, whence the assertion.

Finally, let A be an arbitrary subset of $\Delta(U)$. Then A can be extended by a μ -nullset to a G_δ set $A^* \subset \Delta(U)$ because $\Delta(U)$ is itself a G_δ . We obtain the missing inequality ‘ \leq ’ as follows:

$$\begin{aligned} \widehat{R}_{K\mu}^A &\leq \widehat{R}_{K\mu}^{A^*} = \int \widehat{R}_{K(.,Y)}^{A^*} d\mu(Y) = \int K(.,Y) 1_{A^*}(Y) d\mu(Y) \\ &= \int^* K(.,Y) 1_A(Y) d\mu(Y) = \int^* \widehat{R}_{K(.,Y)}^A d\mu(Y) \end{aligned}$$

according to Proposition 3.9. We have thus shown that (3.1) holds for any set $A \subset \Delta(U)$. According to the last equality the upper integral in the above display becomes a true integral if the subset A of $\Delta(U)$ is μ -measurable.

The general case of the theorem. By Propositions 2.9 and 3.9 we have

$$\widehat{R}_{K\mu}^A = \widehat{R}_{K\mu}^{A \cap \Delta(U)} + \widehat{R}_v^{A \cap U} = \int^* \widehat{R}_{K(.,Y)}^{A \cap \Delta(U)} d\mu(Y) + \widehat{R}_v^{A \cap U},$$

where

$$\begin{aligned} v &:= K\mu - \widehat{R}_{K\mu}^{A \cap \Delta(U)} = \int K(.,Y) d\mu(Y) - \int^* \widehat{R}_{K(.,Y)}^{A \cap \Delta(U)} d\mu(Y) \\ &= \int_* K(.,Y) (1 - 1_{A^* \cap \Delta_1(U)}(Y)) d\mu(Y) = K\lambda, \end{aligned}$$

with $\lambda := (1 - 1_{A^* \cap \Delta_1(U)})\mu$, A^* denoting again a G_δ set containing A such that $\mu^*(A^* \setminus A) = 0$. Since $\lambda \leq \mu$, λ is carried by $\Delta_1(U)$. It follows that

$$\widehat{R}_v^{A \cap U} = \int \widehat{R}_{K(.,Y)}^{A \cap U} d\lambda(Y) = \int \widehat{R}_{K(.,Y)}^{A \cap U} (1 - 1_{A^* \cap \Delta_1(U)}(Y)) d\mu(Y),$$

$$(3.3) \quad \widehat{R}_{K\mu}^A = \int (K(.,Y) 1_{A^* \cap \Delta_1(U)}(Y) + \widehat{R}_{K(.,Y)}^{A \cap U} (1 - 1_{A^* \cap \Delta_1(U)}(Y))) d\mu(Y).$$

Similarly, for any $Y \in \Delta_1(U)$,

$$\widehat{R}_{K(.,Y)}^A = \widehat{R}_{K(.,Y)}^{A \cap \Delta(U)} + \widehat{R}_{v_Y}^{A \cap U} = 1_{A \cap \Delta_1(U)}(Y)K(.,Y) + \widehat{R}_{v_Y}^{A \cap U},$$

$$v_Y := K(.,Y) - \widehat{R}_{K(.,Y)}^{A \cap \Delta(U)} = (1 - 1_{A \cap \Delta_1(U)}(Y))K(.,Y) = K\lambda_Y$$

with $\lambda_Y := (1 - 1_{A^* \cap \Delta_1(U)}(Y))\mu$ carried by $\Delta_1(U)$. It follows that

$$\widehat{R}_{v_Y}^{A \cap U} = \int \widehat{R}_{K(.,Z)}^{A \cap U} d\lambda_Y(Z) = (1 - 1_{A \cap \Delta_1(U)}(Y))\widehat{R}_{K\mu}^{A \cap U},$$

$$\widehat{R}_{K(.,Y)}^A = 1_{A \cap \Delta_1(U)}(Y)K(.,Y) + (1 - 1_{A \cap \Delta_1(U)}(Y))\widehat{R}_{K\mu}^{A \cap U}.$$

The upper integral of this expression for $\widehat{R}_{K(.,Y)}^A$ with respect to $d\mu(Y)$ is just the right hand member of (3.3) because $1_{A \cap \Delta_1(U)} = 1_{A^* \cap \Delta_1(U)}$ μ -a.e. This proves that indeed $\widehat{R}_{K\mu}^A = \int^* \widehat{R}_{K(.,Y)}^A d\mu(Y)$. \square

Corollary 3.11. *Let $u \in \mathcal{S}(U)$, let $A \subset \Delta(U)$, and let μ denote the (unique) representing measure for the invariant function \widehat{R}_u^A carried by $\Delta_1(U)$. Then μ is carried by A (in the sense that $\mu^*(\mathbb{C}A) = 0$) if and only if $\widehat{R}_u^A = u$.*

Proof. By the above theorem together with Proposition 3.9 we have

$$\widehat{R}_u^A = \int^* \widehat{R}_{K(.,Y)}^A d\mu(Y) = \int 1_{A^*}(Y)K(.,Y)d\mu(Y) = K(1_{A^*}\mu),$$

and by uniqueness this equals $u = K\mu$ if and only if $1_{A^*}\mu = \mu$, which means that μ shall be carried by A . \square

The above corollary, sharpening Proposition 2.13 (in the present setting), is an analogue of [6, Proposition 3.13], where μ is carried only by the closure \overline{A} of A , \overline{A} supposed to be contained in U .

In the following minimal-fine boundary minimum property the requirement $Y \in \Delta_1(U)$ (rather than $Y \in \Delta(U)$) is motivated by the fact that $\Delta_1(U)$ is the set of all mf-limit points of U in \overline{U} , as noted after Definition 3.4, and so the stated mf-lim inf $_{x \rightarrow Y}$ is defined for $Y \in \Delta_1(U)$ only.

Proposition 3.12. *Let u be finely superharmonic on U , and suppose that*

$$\text{mf-lim inf}_{x \rightarrow Y, x \in U} u(x) \geq 0 \quad \text{for every } Y \in \Delta_1(U).$$

If moreover $u \geq -s$ on U for some $s \in \mathcal{S}(U)$ then $u \geq 0$ on U .

Proof. For given $\varepsilon > 0$ and $Y \in \Delta(U)$ there exists by the assumed boundary inequality an mf-fine open mf-neighborhood $W_Y \subset \overline{U}$ of Y such that $u > -\varepsilon$ on $W_Y \cap U$. (Take for example $W_Y = \{Y\}$ for $Y \in \Delta(U) \setminus \Delta_1(U)$, cf. Definition 3.4 (c).) In terms of the mf-open set $W := \bigcup \{W_Y : Y \in \Delta(U)\}$ containing $\Delta(U)$ we infer that $u > -\varepsilon$ on $W \cap U = \bigcup \{W_Y \cap U : Y \in \Delta(U)\}$. The set

$E := \overline{U} \setminus W$ is mf-closed and contained in U , and so E is finely closed, as noted before Proposition 3.5. Furthermore, E is minimal-thin at Y in view of Definition 3.4 (b), and hence $\widehat{R}_{K(.,Y)}^E$ is a fine potential on U for each $Y \in \Delta(U)$. It follows that \widehat{R}_s^E likewise is a fine potential. To see this, write $s = K\sigma$ and $\widehat{R}_{K(.,Y)}^E = K\lambda_Y$ with unique representing measures σ on $U \cup \Delta(U)$ and λ_Y on U , respectively, cf. [6, Corollary 3.25]. By Lemma [6, Lemma 3.21] we have

$$\begin{aligned} \widehat{R}_s^E &= \int_{U \cup \Delta_1(U)} \widehat{R}_{K(.,Y)}^E d\sigma(Y) = \int_{U \cup \Delta_1(U)} K\lambda_Y d\sigma(Y) \\ &= \int_{U \cup \Delta_1(U)} \left(\int_U K(.,z) d\lambda_Y(z) \right) d\sigma(Y) = \int_U K(.,z) d\nu(z), \end{aligned}$$

where the measure $\nu = \int_{U \cup \Delta_1(U)} \lambda_Y d\sigma(Y)$ on U is the integral with respect to σ of the family $(\lambda_Y)_{Y \in U \cup \Delta_1(U)}$ of measures on U , cf. [3, 3, proposition 1]. In particular, ν is carried by U along with each λ_Y . Hence \widehat{R}_s^E is indeed a fine potential according to [6, Corollary 3.26]. The (possibly empty) fine interior V of E has relative fine boundary $U \cap \partial_f V \subset U \setminus V \subset U \cap \widetilde{W}$. We have $u + \varepsilon \geq 0$ on $U \cap W$ and hence by fine continuity on $U \cap \widetilde{W} \supset U \cap \partial_f V$. Furthermore, $u + \widehat{R}_s^E = u + s$ on $U \cap b(E) \supset V$, where $b(E)$ denotes the base of E in Ω ; and so $u + \varepsilon \geq -\widehat{R}_s^E$ on V . Altogether, it follows by the relative fine boundary minimum property [8, Theorem 10.8] applied to the fine potential $p = \widehat{R}_s^E$ that $u + \varepsilon \geq 0$ on V . As noted above, the same inequality holds on $U \cap \widetilde{W} \supset U \setminus V$ and thus on all of U . By varying ε we conclude that indeed $u \geq 0$ on all of U . \square

In view of Proposition 3.5 we have the following weaker Martin boundary minimum property relative to the natural topology on \overline{U} . Both properties are used in [7].

Corollary 3.13. *Let u be finely superharmonic on U , and suppose that*

$$\liminf_{x \rightarrow Y, x \in U} u(x) \geq 0 \quad \text{for any } Y \in \Delta(U).$$

If moreover u is lower bounded then $u \geq 0$ on U .

We proceed to define sweeping on subsets of \overline{U} relative to the minimal-fine topology, and to show that sweeping on A relative to the mf-topology coincides with sweeping on A relative to the natural topology as defined in Definition 2.4.

Definition 3.14. Let $A \subset \overline{U}$. For any function $u \in \mathcal{S}(U)$ the reduction of u on A relative to the mf-topology is defined by

$${}^1R_u^A = \inf\{v \in \mathcal{S}(U) : v \geq u \text{ on } A \cap U \text{ and on } W \cap U \text{ for some } W \in {}^1\mathcal{W}(A)\},$$

where ${}^1\mathcal{W}(A)$ denotes the family of all mf-open sets $W \subset \overline{U}$ such that $W \supset A \cap \Delta(U)$. The sweeping of u on A is defined as the greatest finely l.s.c. minorant of ${}^1R_u^A$ and is denoted by ${}^1\widehat{R}_u^A$.

The function ${}^1\widehat{R}_u^A$ is of class $\mathcal{S}(U)$. Similarly to reduction and sweeping relative to the natural topology we have

$${}^1R_u^A = \inf\{R_u^{(A \cup W) \cap U} : W \in {}^1\mathcal{W}(A)\},$$

$${}^1\widehat{R}_u^A = \widehat{\inf}\{\widehat{R}_u^{(A \cup W) \cap U} : W \in {}^1\mathcal{W}(A)\}.$$

Furthermore, there is a decreasing sequence (W_j) of sets $W_j \in {}^1\mathcal{W}(A)$ (depending on u) such that it suffices to take for W the sets W_j , in the above definitions and alternative expressions (this is shown in the same way as in the case of sweeping relative to the natural topology by application of the fundamental convergence theorem and the quasi-Lindelöf property for finely u.s.c. functions). For any subset A of U , the present reduction ${}^1R_u^A$ and sweeping ${}^1\widehat{R}_u^A$ on A relative to \overline{U} clearly reduce to the usual reduction and sweeping on A relative to U . Since the mf-topology is finer than the natural topology (Proposition 3.5), we clearly have ${}^1R_u^A \leq R_u^A$ and ${}^1\widehat{R}_u^A \leq \widehat{R}_u^A$.

As in the Euclidean case [2, Theorem 8.3.1] we have the following

Corollary 3.15. *For any $u \in \mathcal{S}(U)$ we have ${}^1\widehat{R}_u^{\Delta(U) \setminus \Delta_1(U)} = 0$.*

Proof. According to Definition 3.4 (c) the set $A := \Delta(U) \setminus \Delta_1(U)$ belongs to ${}^1\mathcal{W}(A)$, and $A \cap U = \emptyset$, whence ${}^1\widehat{R}_u^A = 0$. \square

We shall need the following analogue of Proposition 3.9:

Lemma 3.16. *For any $A \subset \Delta(U)$ and $Y \in U \cup \Delta_1(U)$ we have ${}^1\widehat{R}_{K(.,Y)}^A = K(.,Y)$ if $Y \in A$, and ${}^1\widehat{R}_{K(.,Y)}^A = 0$ if $Y \notin A$.*

Proof. If $Y \notin A$ then ${}^1\widehat{R}_{K(.,Y)}^A \leq \widehat{R}_{K(.,Y)}^A = 0$ by Proposition 2.14 (ii). If $Y \in A$ and hence $Y \notin U$, then $Y \in \Delta_1(U)$, and ${}^1\widehat{R}_{K(.,Y)}^A = K(.,Y)$ because we even have ${}^1\widehat{R}_{K(.,Y)}^{\{Y\}} = K(.,Y)$. In fact, for any $W \in {}^1\mathcal{W}(\{Y\})$,

$$K(.,Y) = \widehat{R}_{K(.,Y)}^U \leq \widehat{R}_{K(.,Y)}^{U \cap W} + \widehat{R}_{K(.,Y)}^{U \setminus W},$$

where the latter term on the right is a fine potential on U by Definition 3.2, $U \setminus W$ being minimal-thin at Y in view of Definition 3.4 (b). By the Riesz decomposition property we obtain $K(.,Y) = u + v$ with $u \leq \widehat{R}_{K(.,Y)}^{U \cap W}$ and $v \leq \widehat{R}_{K(.,Y)}^{U \setminus W}$. This shows that $v \preccurlyeq K(.,Y)$ and hence $v = 0$, v being a fine potential along with $\widehat{R}_{K(.,Y)}^{U \setminus W}$, and $K(.,Y)$ being invariant since $Y \in \Delta_1(U)$. Thus $K(.,Y) = u \leq \widehat{R}_{K(.,Y)}^{U \cap W}$, obviously with equality. By varying W we infer by Definition 3.14 that indeed ${}^1\widehat{R}_{K(.,Y)}^{\{Y\}} = K(.,Y)$. \square

The six assertions of Proposition 2.6 carry over along with their proofs when reductions and sweepings are taken with respect to the minimal-fine topology on U instead of the smaller natural topology, and of course $\mathcal{W}(A)$ is replaced by ${}^1\mathcal{W}(A)$ for any $A \subset \overline{U}$. The same applies to Propositions 2.7, 2.8, and 2.9.

Theorem 3.17. *Let $A \subset \overline{U}$ and $u \in \mathcal{S}(U)$. Then ${}^1\widehat{R}_u^A = \widehat{R}_u^A$.*

Proof. This is obvious if $A \subset U$. Next, for $A \subset \Delta(U)$, write $u = K\mu$ with μ carried by $U \cup \Delta_1(U)$. For any $W \in {}^1\mathcal{W}(A)$ we have by [6, Lemma 3.21]

$$\begin{aligned} \widehat{R}_{K\mu}^{W \cap U} &= \int \widehat{R}_{K(\cdot, Y)}^{W \cap U} d\mu(Y) \geq \int^* {}^1\widehat{R}_{K(\cdot, Y)}^A d\mu(Y) \\ &= \int^* \widehat{R}_{K(\cdot, Y)}^A d\mu(Y) = \widehat{R}_{K\mu}^A, \end{aligned}$$

the second equality because ${}^1\widehat{R}_{K(\cdot, Y)}^A = \widehat{R}_{K(\cdot, Y)}^A = 1_A(Y)K(\cdot, Y)$ for $Y \in U \cup \Delta_1(U)$ according to Lemma 3.16 and Proposition 3.9, respectively; and the third equality follows by Theorem 3.10. By varying $W \in {}^1\mathcal{W}(A)$ this yields ${}^1\widehat{R}_{K\mu}^A \geq \widehat{R}_{K\mu}^A$, actually with equality. It follows by Proposition 2.9 and its mf version that indeed ${}^1\widehat{R}_u^A = \widehat{R}_u^A$ for any $A \subset \overline{U}$ because v is the same in either case (by what has just been shown), and hence ${}^1\widehat{R}_v^{A \cap U} = \widehat{R}_v^{A \cap U}$ since $A \cap U \subset U$. \square

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